Algebraic approaches to things

In late 1993, the U.S. Department of Education Office of Research sponsored a colloquium as a first step in a major effort, the Algebra Initiative, that will rethink the importance of algebra and algebraic thinking from kindergarten through graduate school. The charge for the colloquium begins with motto: **``Algebra is the language of mathematics.''**

Algebraists like a good calculation

Underneath it all, algebra is the study of sets equipped with one or more binary operations. The spirit of algebra is the study of how to reason about the behavior of these binary operations. A set equipped with binary operations is a system in which one can calculate, and algebra asks the question, ``What are the rules for calculating in this system?'' The calculations can be with numbers, abstract symbols, functions, propositions, permutations, even calculations. Sometimes the calculations are just for fun, as in the famous:

$$\left(\frac{\frac{1}{x^2}+\frac{1}{y^2}}{\frac{1}{x^2}-\frac{1}{y^2}} - \frac{\frac{1}{x^2}-\frac{1}{y^2}}{\frac{1}{x^2}+\frac{1}{y^2}} \right) \div \left(\frac{8}{\left(\frac{x+y}{x-y}+\frac{x-y}{x+y}\right)\left(\frac{x^2}{y^2}+\frac{y^2}{x^2}-2\right)} \right)$$

Algebraists use abstraction

The word ``abstract'' has taken on negative connotations in the mathematics education community, where it is often used as an opposite to ``concrete'' or even ``simple'' or ``clear.'' In algebra (and in many other parts of mathematics), abstraction is a natural and powerful tool for expressing ideas and obtaining new insights and results.

Algebraists like algorithms

Algebra began as a search for algorithms for solving equations, and algebra has never lost its taste for finding recipes for solving classes of problems. Algebraic algorithms come in all sorts. Some provide shortcuts for calculations that could, in principle, be carried out. Others tell you about properties of algebraic objects that would be quite difficult to determine without the algorithms. Most have the characteristic that, if you're not in on the process of designing them, they seem quite astounding. On the other hand, for the designer of an algorithm, the finished product is often the result of capturing the essence of extensive calculations. Here are some examples of how algorithms are used in algebra and how algebraic algorithms are applied outside algebra:

Algebraists break things into parts

A useful technique in algebra is to identify the ``building blocks'' of a structure. Algebraists like ``structure theorems'' (or ``decomposition theorems'') that usually say something like ``every object under consideration is a combination of a collection of very simple objects.'' The most famous decomposition theorem is the fundamental theorem of arithmetic: every integer except for 0, 1, and -1 can written (in essentially one way) as a product of primes.

Algebraists extend things

The calculations, algorithms, and decompositions described above all take place in algebraic systems (sets of things that are equipped with binary operations that allow you to calculate). New insights come when you see how a calculation or theorem behaves when you put a given system inside a larger one.

Algebraists represent things

There are formal mathematical definitions of representations, in which the elements of one algebraic structure correspond to certain functions on another, but we adopt a broader and more informal use of the word here. Essentially, the idea is to use a well-understood structure to study a less well known one or to set up an interplay between seemingly different structures that proves fruitful in the study of both.